UNIT - I

BANACH SPACES

Linear Space:- A linear space or a vector space is an additive abelian group with the property that any vector 'x' and a scalar 'α' can be combined by scalar multiplication to give a vector αx in such a manner that

i, $\alpha(x+y) = \alpha x + \alpha y$

ii, $(\alpha+\beta)x = \alpha x + \beta x$

iii, $(\alpha\beta)x = \alpha(\beta x)$

iv, $1 \cdot x = x \cdot 1 = x$

Note:- A linear space is a real linear space or a complex linear space according as the scalars are real numbers or complex numbers.

Normed linear space:- A normed linear space is a linear space in which to each vector 'x' their corresponds a real number $\|x\|$ is called "Norm of x" in such a way that

i, $\|x\| \geq 0$

ii, $\|x\| = 0$ iff $x = 0$

iii, $\|x+y\| \leq \|x\| + \|y\|$

iv, $\|\alpha x\| = |\alpha| \|x\|$

Note:- 1, The non-negative (linear) or real number $\|x\|$ is treated as the length of the vector 'x'

2, If we consider $\|x\|$ as a real function defined on 'n' then this function is called as norm.

Result:- 1. A normed linear space 'N' is a metric space with respect to the metric given by $d(x,y) = \|x-y\|$

In order to prove that N is a metric space we prove the following three conditions.

i, $d(x,y) \geq 0$ and $d(x,y) = 0 \Leftrightarrow x = y$

ii, $d(x,y) = d(y,x)$

iii, $d(x,z) \leq d(x,y) + d(y,z)$

To Prove I :-

Given $d(x,y) = \|x-y\|$

Since norm is always non-negative we have $\|x-y\| \geq 0$

$\Rightarrow d(x,y) \geq 0$

Now $d(x,y) = 0 \Leftrightarrow \|x-y\| = 0$

$\Leftrightarrow x-y = 0$

$\Leftrightarrow x = y$

To prove II :-

$d(x,y) = \|x-y\|$

$= \|-(y-x)\|$

$= |-1| \|y-x\|$

$= \|y-x\|$

$= d(y,x)$

$\therefore d(x,y) = d(y,x)$

To Prove III :-

$d(x,z) = \|x-z\|$

$= \|x-y+y-z\|$

$\leq \|x-y\| + \|y-z\|$

$\leq d(x,y) + d(y,z)$

$\therefore d(x,z) \leq d(x,y) + d(y,z)$

Complete Normed Linear Space is called a "Banach Space"

Result :- In a normed linear space 'N' we have $|\|x\|-\|y\|| \leq \|x-y\|$

Proof :- Consider $\|x\| = \|x-y+y\|$

$\leq \|x-y\| + \|y\|$

$\Rightarrow \|x\| - \|y\| \leq \|x-y\|$ — (1)

Consider $\|y\| = \|y-x+x\|$

$\leq \|y-x\| + \|x\|$

$\Rightarrow \|y\| - \|x\| \leq \|y-x\|$

$\Rightarrow \|y\| - \|x\| \leq \|x-y\|$

$\Rightarrow -(\|x\| - \|y\|) \leq \|x-y\|$

$\Rightarrow -\|x-y\| \leq \|x\| - \|y\|$ — (2)

From (1) and (2)

$-\|x-y\| \leq \|x\| - \|y\| \leq \|x-y\|$

$\Rightarrow |\|x\| - \|y\|| \leq \|x-y\|$

Result-3 In a normed linear space 'N'

i, The norm is a continuous function.

ii, vector addition and scalar multiplication are jointly contin-
-ous.

Proof:- To prove i :-

Let $\{x_n\}$ be a convergent sequence in the normed linear sp-
-ace 'N'

$\therefore$ By definition, for a given $\epsilon>0$ $\exists$ a +ve integer $N \ni \|x_n - x\|<\epsilon$ $\forall n \geq N$

ie, $\lim_{n\to\infty} x_n = x$

By result-2 we have $\|x_n\| - \|x\| \leq \|x_n - x\|$

$<\epsilon \quad \forall n \geq N$.

$\lim_{n\to\infty} \|x_n\| = \|x\|$

Hence norm is a continuous function.

To prove ii, :-

First we prove that vector addition is continuous.

Let $\{x_n\}$ and $\{y_n\}$ be two convergent sequences in 'N'.

$\therefore$ By definition, for a given $\epsilon>0$ $\exists$ a +ve integer 'N' $\ni \|x_n - x\|<\epsilon$, $\|y_n - y\|<\epsilon$ $\forall n \geq N$

Now we have to prove that $x_n + y_n \to x + y$

Again by result-2 we have $|\|x_n+y_n\| - \|x+y\|| \leq \|(x_n+y_n)-(x+y)\|$

$\leq \|x_n - x\| + \|y_n - y\|$

$< \epsilon/2 + \epsilon/2$

$\therefore \lim_{n\to\infty} \|x_n+y_n\| = \|x+y\|$ $\quad < \epsilon$

Now we prove that scalar multiplication is continuous.

Let $\{\alpha_n\}$ and $\{x_n\}$ be two convergent sequences

ie, $\lim_{n\to\infty} \alpha_n = \alpha$; $\lim_{n\to\infty} x_n = x$

Now we prove that $\alpha_n x_n \to \alpha x$

Again by above result, we have $|\|\alpha_n x_n\| - \|\alpha x\|| \leq \|\alpha_n x_n - \alpha x\|$

$|\|\alpha_n x_n\| - \|\alpha x\|| \leq \|\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x\|$

$= \|\alpha_n(x_n - x) + x(\alpha_n - \alpha)\|$

$\leq \|\alpha_n(x_n - x)\| + \|x(\alpha_n - \alpha)\|$

$= |\alpha_n| \|x_n - x\| + |x| \|\alpha_n - \alpha\|$

$< \epsilon$

$\therefore \lim_{n\to\infty} \alpha_n x_n = \alpha \cdot x$

$\therefore$ vector addition and scalar multiplication are jointly conti-
-nuous.

Hence proved.

Theorem:- If 'M' is a closed linear subspace of a normed li-
-near space 'N' and if norm of the coset $x+M$ in the quoti-
-ent space $\frac{N}{M}$ is given by $\|x+M\| = \inf\{\|x+m\| : m \in M\}$ then $\frac{N}{M}$ is a
normed linear space further more if 'N' is a banach space
then so is $\frac{N}{M}$

Proof:- Given that 'N' is a normed linear space
M is a closed linear subspace of a normed linear space and
$\|x+M\| = \inf\{\|x+m\| : m \in M\}$

Claim:- 1. $\frac{N}{M}$ is a normed linear space

2. $\frac{N}{M}$ is a banach space whenever N is a Banach Space.

To prove 1:- In order to prove this we prove the followin
-g three conditions

i. $\|x+M\| \geq 0$ and $\|x+M\| = 0 \Leftrightarrow x+M = 0+M$

ii. $\|(x+M)+(y+M)\| \leq \|x+M\| + \|y+M\|$

iii. $\|\alpha(x+M)\| = |\alpha| \, \|x+M\|$

(i) Since norm is a non-negative real number
$\Rightarrow \{\|x+m\| : m \in M\}$ is the set of all non-negative real numbers
which is bounded below and hence infimum exists, which
is also a non-negative real number

$\therefore \|x+M\| \geq 0$

Now we prove that $\|x+M\| = 0 \Leftrightarrow x+M = 0+M$

First suppose that $\|x+M\| = 0$

$$\Rightarrow \inf\{\|x+m\| : m \in M\} = 0$$

$\Rightarrow \exists$ a sequence $\{m_k\}$ in M such that $\lim_{k\to\infty} \|x+m_k\| = 0$

$\Rightarrow \lim_{k\to\infty} \|x+m_k\| = 0$

$\Rightarrow \lim_{k\to\infty} m_k = -x$

$\Rightarrow -x \in M$ ($\because$ M is closed)

$\Rightarrow$ "$x \in M$" ($\because$ M is a linear Subspace of N)

Since $x \in M$ and $m \in M \Rightarrow x+m \in M$

$\Rightarrow x+M = M = 0+M$

which is the zero vector in $\frac{N}{M}$

conversely, suppose that $x+M = 0+M = M$

$\Rightarrow x \in M$ and $m \in M$

$\Rightarrow x+m \in M$ where $x+m = 0$

Now $\|x+M\| = \inf\{\|x+m\| : m \in M, x \in M\} = 0$

(Since $\|x+M\| = 0$ is the zero vector and 'M' is linear space)

$\therefore \|x+M\| = 0$

(ii) Consider $\|(x+M)+(y+M)\| = \|(x+y)+M\|$

$= \inf\{\|x+y+m\| : m \in M\}$

$= \inf\{\|x+y+m_1+m_2\| : m_1, m_2 \in M\}$,

where $M = \max\{m_1, m_2\}$

$\leq \inf\{\|x+m_1\| ; m_1 \in M\} + \inf\{\|y+m_2\| : m_2 \in M\}$

$\leq \|x+M\| + \|y+M\|$

$\therefore \|(x+M)+(y+M)\| \leq \|x+M\| + \|y+M\|$

(iii) Consider $\|\alpha(x+M)\| = \inf\{\|\alpha(x+m)\| : m \in M\}$

$= \inf\{|\alpha| \|(x+m)\| : m \in M\}$

$= |\alpha| \inf\{\|x+m\| : m \in M\}$

$= |\alpha| \|x+M\|$

$\therefore \|\alpha(x+M)\| = |\alpha| \|x+M\|$

$\therefore \frac{N}{M}$ is a normed linear space.

To prove 2:- Let $\{S_n+M\}$ is a cauchy sequence in $\frac{N}{M}$ where $S_n \in N$

In order to prove that $\frac{N}{M}$ is complete it is enough to prove that every cauchy sequence has a cgt sub-sequence.

Now we are going to construct a cgt sub-sequence of the above cauchy sequence as follows

Since $\{S_n+M\}$ is a cauchy sequence, by definition for a given $\epsilon = 1/2$ $\exists$ +ve integer n_1 $\ni$ $\|(S_n+M)-(S_m+M)\| < 1/2$ $\forall$ $n \geq n_1$

Set $S_{n_1} = x_1 \in N$

Again for a given $\epsilon = 1/2^2$ $\exists$ +ve integer $n_2 \ni \|(S_n+M) - (S_m+M)\| < \frac{1}{2^2}$ $\forall$ $n \geq n_2$

Set $S_{n_2} = x_2 \in N$

continuing in this way we get some +ve integers $n_1, n_2, n_3, n_4, n_5 \cdots n_k$ and some vectors $x_1, x_2, x_3 \cdots x_k$ such that

that $\|(S_n+M)-(S_m+M)\| < \frac{1}{2^{k+1}}$ $\forall$ $n \geq n_{k+1}$ and $n_{k+1} > n_k$

Set $S_{n_{k+1}} = x_{k+1} \in N$

Thus, we get a subsequence $\{x_k+M\}$ of $\{S_n+M\}$ such that $\|(x_{k+1}+M)-(x_k+M)\| < \frac{1}{2^k}$ $\forall$ $k=1,2,3\ldots\ldots$

Let $y_1 \in x_1+M \Rightarrow y_1 = x_1+m_1$ and $\|(x_1+M)-(x_2+M)\| < \frac{1}{2}$

Now $\Rightarrow \|(x_1-x_2)+M\| < 1/2$

$\Rightarrow \inf\{\|x_1-x_2+m)\| ; m \in M\} < 1/2$

$\Rightarrow \inf\{\|x_1-x_2+m_1-m_1+m\| : m_1, m_2 \in M\} < 1/2$

$\Rightarrow \inf\{\|x_1+m_1-(x_2+m_1-m)\| : m_1, m_2 \in M\} < 1/2$

$\Rightarrow \|y_1-y_2\| < 1/2$

where $y_1 = x_1+m_1$ and $y_2 = x_2+m_1-m$

Thus for a given y_1 in x_1+M $\exists$ y_2 in x_2+M $\ni$ $\|y_1-y_2\| < 1/2$

lly $\exists$ y_3 in x_3+M and $\|y_2-y_3\| < 1/2^2$

Continuing in this way we get a sequence $\{y_n\}$ in N

Now we have to prove that $\{y_n\}$ is a cauchy sequence.

for $m<n$,

consider $\|y_m-y_n\| = \|y_m-y_{m+1}+y_{m+1}+\cdots+y_{n-1}-y_n\}$

$\leq \|y_m-y_{m+1}\| + \|y_{m+2}+y_{m+1}\| + \cdots \|y_{n-1}-y_n\|$

$\leq 1/2^m + \frac{1}{2^{m+1}} + \cdots + 1/2^{n-1}$

$= \frac{1}{2^m}(1+1/2+\cdots)$

$= \frac{1}{2^m}\left(\frac{1}{1-1/2}\right) = \frac{1}{2^m}\left(\frac{1}{1/2}\right) = \frac{2}{2^m}$

$= \frac{1}{2^{m-1}} \to 0$ as $m \to \infty$

$\therefore \{y_n\}$ is a cauchy sequence in N where N is a complete normed linear space.

$\Rightarrow$ This cauchy sequence $\{y_n\}$ cgs to some point 'y' in N

ie, $\|y_n-y\| \to 0$ as $n \to \infty$ —(1)

Now we prove that our subsequence $\{x_k+M\} \to y+M$

consider $\|x_k + M - (y+M)\| = \|(x_k - y) + M\|$

$= \inf\{\|x_k - y + m\|;\ m \in M\}$

$\leq \|x_k - y + m\|$

$= \|x_k + m - y\|$

$= \|y_k - y\| \to 0$ as $k \to \infty$ ($\because$ By eqn ①)

$\therefore \|(x_k + M) - (y+M)\| \to 0$ as $k \to \infty$

$\Rightarrow x_k + M$ cgs to $y + M$

$\Rightarrow \frac{N}{M}$ is a completely normed linear space and hence a banach space.

Ex:- The space $\mathbb{C}^n$ of all n-tuples is a normed linear space with the norm is defined by $\|x\| = \sum_{i=1}^{n} |x_i^2|^{1/2}$ and is a banach space whenever 'C' is a banach space.

Proof:- To prove $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$:-

Since $\|x\| = \sum_{i=1}^{n} |x_i^2|^{1/2}$, where modulus is always non-negative

$\Rightarrow |x_i^2| \geq 0$ for all $i = 1, 2, 3 \dots n$

$\Rightarrow \sum_{i=1}^{n} |x_i^2| \geq 0$

$\Rightarrow \sum_{i=1}^{n} |x_i^2|^{1/2} \geq 0$

$\Rightarrow \|x\| \geq 0$

Let $\|x\| = 0$

$\Leftrightarrow \sum_{i=1}^{n} |x_i^2|^{1/2} = 0$

$\Leftrightarrow \sum_{i=1}^{n} |x_i^2| = 0$

$\Leftrightarrow \sum_{i=1}^{n} |x_i| = 0$

$\Leftrightarrow |x_i| = 0 \ \forall i = 1, 2, 3 \dots n$

$\Leftrightarrow x = (x_1, x_2, x_3 \dots x_n) = 0$

To prove $\|x+y\| \leq \|x\| + \|y\|$:-

consider, $\|x+y\| = \sum_{i=1}^{n} |(x_i + y_i)^2|^{1/2}$

$\|x+y\|^2 = \sum_{i=1}^{n} |(x_i + y_i)|\,|x_i + y_i|$

$\leq \sum_{i=1}^{n} |x_i + y_i|\,|x_i| + \sum_{i=1}^{n} |x_i + y_i|\,|y_i|$

$= \|x+y\|\,\|x\| + \|x+y\|\,\|y\|$

$\|x+y\|^2 \leq \|x+y\| (\|x\| + \|y\|)$

Case i) :- If $\|x+y\| = 0$ then our claim is trivially true.

Case ii) :- If $\|x+y\| \neq 0$ then $\|x+y\| \leq \|x+y\|/\|\,(\|x\|+\|y\|)$

$\|x+y\| \leq \|x\|+\|y\|$

To Prove $\|\alpha x\| \leq |\alpha|\,\|x\|$:-

Consider $\|\alpha x\| = \sum_{i=1}^{n} |\alpha x_i|^{2\,1/2}$

$= \sum_{i=1}^{n}(|\alpha|^2 |x_i^2|)^{1/2}$

$= |\alpha| \sum_{i=1}^{n} |x_i^2|^{1/2} = \|x\|\,|\alpha|$

$\therefore \mathbb{C}^n$ is a normed linear space.

To prove $\mathbb{C}^n$ is a banach space:-

Let $(x^{(1)}, x^{(2)}, \ldots x^{(n)} \ldots)$ is a cauchy sequence in $\mathbb{C}^n$ in which rea and every individual term is an n-tuple of complex numbers

ie, $x_1^{(1)}, x_2^{(1)} \ldots, x_n^{(1)} = x^{(1)}$

$\vdots$

$x^{(n)} = x_1^{(n)}, x_2^{(n)}, x_3^{(n)} \ldots x_n^{(n)}$

Now by definition, for a given $\epsilon > 0$ $\exists$ a +ve integer $n_0 \ni \|x^{(l)} - x^{(m)}\| < \epsilon$

$l, m \geq n_0$

$\Rightarrow \left[\sum_{i=1}^{n} (x_i^l - x_i^m)^2\right]^{1/2} < \epsilon \;\; \forall\, l, m \geq n_0$

$\Rightarrow \sum_{i=1}^{n} |(x_i^l - x_i^m)^2|^{1/2} < \epsilon \;\; \forall\, l, m \geq n_0$

$\Rightarrow |x_i^l - x_i^m| < \epsilon \;\; \forall\, l, m \geq n_0$ —(1)

$\therefore \{x_i^l\}$ is a cauchy sequence in '$\mathbb{C}$' where '$\mathbb{C}$' is a completely -rmed linear space.

$\Rightarrow x_i^l$ cgs to some point a_i in '$\mathbb{C}$' $\forall\, i = 1, 2, 3 \ldots n$

Now we prove that our cauchy sequence $(x^{(1)}, x^{(2)}, \ldots x^{(n)})$ cg to $a = a_1, a_2, a_3 \ldots a_n$:-

Fixing 'm' and letting 'l' $\to \infty$ in eqn (1). we get

$|x_i^m - a_i| < \epsilon \;\; \forall\, i = 1, 2, 3 \ldots n$

$\Rightarrow \sum_{i=1}^{n} |(x_i^m - a_i)^2| < \epsilon$

$\Rightarrow \sum_{i=1}^{n} |(x_i^m - a_i)^2|^{1/2} < \epsilon$

$\Rightarrow \|x^m - a\| < \epsilon$

$\therefore x^m \to a$ $\therefore \mathbb{C}^n$ is complete and also a normed linear space hence is a banach space

Ex:-2 'R' and 'C' are the smallest normed linear spaces with the norm $\|x\| = |x|$ and they are also banach spaces.

Ex:-3 The space $\beta(x)$ of all continuous bounded scalar valued functions of the topological spaces X is a normed linear space with respect to the norm $\|f\| = \sup\{|f(x)| : x \in X\}$ also a banach space.

Problems:-

1. Let 'N' be a normed linear space and $S = \{x / \|x\| = 1\}$ is the subset of 'N' then N is a banach space $\Leftrightarrow$ 'S' is complete

Part:-1 :- Suppose that 'N' is a banach space.

In order to prove that 'S' is complete.

Let us consider a cauchy sequence $\{x_n\}$ in 'S'

Since $\{x_n\} \in S \Rightarrow \|x_n\| = 1$

Since $\{x_n\} \in S$ and 'S' is a subset of N

$\Rightarrow \{x_n\} \in N$, where N is a banach space.

The cauchy sequence $\{x_n\}$ cgs to some point 'x' in N.

ie, $\lim_{n\to\infty} \{x_n\} = x$

$\Rightarrow \|\lim_{n\to\infty} x_n\| = \|x\|$

$\Rightarrow \lim_{n\to\infty} \|x_n\| = \|x\|$

$\Rightarrow \lim_{n\to\infty} 1 = \|x\|$

$\Rightarrow 1 = \|x\|$

$\Rightarrow x \in S$

Part:-2 :- Conversely, suppose that 'S' is complete.

Claim:- 'N' is a banach space

Let $\{y_n\}$ be a cauchy sequence in N

For a given $\epsilon > 0$ $\exists$ a +ve integer n_0 such that $\|y_n - y_0\| < \epsilon$ $\forall$ $n > n_0$

Let $x_n = \dfrac{y_n}{\|y_n\|}$

$\Rightarrow \|x_n\| = \left\|\dfrac{y_n}{\|y_n\|}\right\| = \dfrac{\|y_n\|}{\|y_n\|} = 1$

$\Rightarrow \therefore x_n \in S$

Now consider, $\|x_m - x_n\| = \left\|\dfrac{y_m}{\|y_m\|} - \dfrac{y_n}{\|y_n\|}\right\|$

metric space N'. In the sense that if $x_n \to x$ in $N \Rightarrow$ $T(x_n) \to T(x)$ in N'

Theorem:- A Let N and N' be two normed linear spaces with the same scalars and let 'T' be a linear transformation of N into N' then the following conditions are equivalent to one another

i, T is continuous.

ii, T is continuous at the origin in the sense that if $x_n \to 0$ $T(x_n) \to 0$

iii, $\exists$ a real number 'k' ≥ 0 with the property that $\|T(x)\| \leq k\|x\|$

iv, If $S = \{x / \|x\| \leq 1\}$ is closed unit sphere in N then $T(S)$ is a bounded subset of N'

Proof:- Given that $T: N \to N'$ be a linear transformation

claim:- i $\Leftrightarrow$ ii $\Leftrightarrow$ iii $\Leftrightarrow$ iv

To prove i $\Leftrightarrow$ ii :-

Assume i, ie, T is continuous

Since T is continuous

By def we have if $x_n \to x$ in N then $T(x_n) \to T(x)$ in N' — (1)

Let us assume that $x_n \to 0 \Rightarrow T(x_n) \to T(0)$ (By eqn (1))

$\Rightarrow T(x_n) \to 0$

$\therefore T$ is continuous at the origin.

Assume ii ie, if $x_n \to 0$ then $T(x_n) \to 0$

Let $\{x_n\}$ be a sequence in N $\ni$ $x_n \to x$ in N

$\Rightarrow x_n - x \to 0$

$\Rightarrow T(x_n - x) \to 0$

$\Rightarrow T(x_n) - T(x) \to 0$

$\Rightarrow T(x_n) \to T(x)$

$\therefore T$ is continuous.

To prove ii $\Leftrightarrow$ iii :-

Assume ii

Suppose that $\exists$ no real number 'k' ≥ 0 with the property that $\|T(x)\| \leq k\|x\|$.

ie, $\exists$ a number 'n' $\ni$ $\|T(x_n)\| > n\|x_n\|$ for all vectors x_n in N

$\Rightarrow \dfrac{\|T(x_n)\|}{n\|x_n\|} > 1$

$\Rightarrow T\left(\frac{\|x_n\|}{n\|x_n\|}\right) > 1$ ($\because T$ is a linear transformation)

$\Rightarrow T(\|y_n\|) > 1$, where $y_n = \frac{x_n}{n\|x_n\|}$

$\Rightarrow \|y_n\| = \left\|\frac{x_n}{n\|x_n\|}\right\|$

$= \frac{\|x_n\|}{n\|x_n\|} = \frac{1}{n} \to 0$ as $n \to \infty$

By our assumption we have, $T(y_n) \to 0$

But we have $T\|y_n\| > 1$

which is a contradiction to our assumption.

$\therefore \exists$ a real number $k \geq 0 \ni \|T(x)\| \leq k\|x\|$

Assume (ii)

Let $x_n \to 0$

$\Rightarrow \|x_n\| \to \|0\| = 0$

$\Rightarrow \|x_n\| \to 0$

By our assumption we have $\|T(x_n)\| \leq K\|x_n\|$

$= K(0)$

$= 0$

$\therefore \|T(x_n)\| = 0$

$\therefore T(x_n) \to 0$ whenever $x_n \to 0$

To prove (ii) $\Leftrightarrow$ iv :-

Assume (ii)

we have given that $S = \{x \mid \|x\| \leq 1\}$ is a closed unit sphere in N

By our assumption, we have $\|T(x)\| \leq K\|x\| \; \forall x \in N$

$\Rightarrow \|T(x)\| \leq K(1) \; \forall x \in S$

$\Rightarrow \|T(S)\| \leq K$

and hence $T(S)$ is a bounded subset of N'

Assume iv

ie, if $S = \{x \mid \|x\| \leq 1\}$ is a closed unit sphere in N then $T(S)$ is a bounded subset of N'

claim:- $\|T(x)\| \leq K\|x\|$

case-i :- if $x = 0$ then our claim is trivially true.

case ii :- if $x \neq 0$ then let $\frac{x}{\|x\|} \in S$

since $T(S)$ is a bounded subset of N'

By def, we have $\|T(S)\| \leq K$ for some constant 'K'

$\|T(\frac{x}{\|x\|})\| = \frac{\|T(x)\|}{\|x\|}$ (1)

$= \frac{1}{\|x\|}\|T(x)\|$

$\leq K$

$\|T(x)\| \leq K\|x\|$

Hence proved.

Definition:- Let N and N' be two normed linear spaces with the same scalars and 'T' is a linear transformation from N into N' if $\exists$ a real number $K>0$ with the property that $\|T(x)\| \leq K\|x\|$ then 'K' is called bound for 'T' and 'T' is called a "bounded linear transformation"

Note:- By above theorem, we have T is continuous $\Leftrightarrow$ T is bounded.

Definition:- If 'T' is a continuous linear transformation of a normed linear space N into a normed linear space N' then the norm of T is defined by $\|T\| = \sup\{\|T(x)\| / \|x\| \leq 1\}$

Note:- 1, If $N \neq \{0\}$ then the $\|T\|$ is defined by $\|T\| = \sup\{\|T(x)\| / \|x\| = 1\}$..

2, The norm is also defined as $\|T\| = \inf\{K / K \geq 0$ and $\|T(x)\| \leq K\|x\|\}$

3, we have $\|T(x)\| \leq \|T\| \, \|x\|$

4, The set of all continuous linear transformations of a normed linear space N into a normed linear space N' is denoted by $\mathcal{B}(N, N')$ (or) $\mathcal{B}(N, N')$

Definition:- A continuous linear transformation of 'N' into itself is called an operator of 'N'.

Note:- The set of all operators on N is denoted by $\mathcal{B}(N)$

Theorem:- B Let N and N' be two normed linear spaces. Then the set $\mathcal{B}(N, N')$ is itself a normed linear space with respect to point-wise linear operations and the norm is defined by $\|T\| = \sup\{\|T(x)\| / \|x\| \leq 1\}$. Further more if N' is a banach space then so is $\mathcal{B}(N, N')$

Proof:- Given that N and N' be two normed linear spaces, $\mathcal{B}(N,N')$ is the set of all continuous linear transformations.

$\|T\| = \sup\{\|T(x)\| : \|x\| \leq 1\}$ (2)

In order to prove that $\mathcal{B}(N,N')$ is a Normed linear space we first prove that it is a linear space.

Let $T_1, T_2 \in \mathcal{B}(N,N')$

$\Rightarrow \|T_1(x)\| \leq k_1\|x\|$ & $\|T_2(x)\| \leq k_2\|x\|$, $\forall x \in N$ & k_1, k_2 are non-negative real numbers

for any scalars α, β

consider $\|(\alpha T_1 + \beta T_2)(x)\| = \|(\alpha T_1)(x) + (\beta T_2)(x)\|$

$$= \|\alpha(T_1 x) + \beta(T_2 x)\|$$

$$\leq \|\alpha T_1(x)\| + \|\beta T_2(x)\|$$

$$\leq |\alpha|\,\|T_1(x)\| + |\beta|\,\|T_2(x)\|$$

$$\leq |\alpha| k_1\|x\| + |\beta| k_2\|x\|$$

$$\leq (|\alpha| k_1 + |\beta| k_2)\|x\|$$

$$\leq k\|x\| \text{ where } k = |\alpha| k_1 + |\beta| k_2$$

$\therefore \alpha T_1 + \beta T_2 \in \mathcal{B}(N,N')$ and hence $\mathcal{B}(N,N')$ is a linear space

Now we prove that $\mathcal{B}(N,N')$ is a normed linear space.

i, $\|T\| \geq 0$ & $\|T\| = 0 \Leftrightarrow T = 0$:-

Since $\|T\| = \sup\{\|T(x)\| : \|x\| \leq 1\}$

where, the supremum of non-negative real numbers is also non-negative $\Rightarrow \|T\| \geq 0$.

and $\|T\| = 0 \Leftrightarrow \sup\{\|T(x)\| : \|x\| \leq 1\} = 0$

$$\Leftrightarrow \|T(x)\| = 0 \text{ and } \|x\| \leq 1$$

$$\Leftrightarrow Tx = 0 \text{ \& } \|x\| \leq 1$$

$$\Leftrightarrow Tx = 0 \text{ \& } x \neq 0$$

$$\Leftrightarrow T = 0$$

ii, To prove $\|(T_1 + T_2)\| \leq \|T_1\| + \|T_2\|$:-

consider $\|T_1 + T_2\| = \sup\{\|(T_1 + T_2)(x)\| : \|x\| \leq 1\}$

$$= \sup\{\|T_1(x) + T_2(x)\| : \|x\| \leq 1\}$$

$$\leq \sup\{\|T_1(x)\| + \|T_2(x)\| : \|x\| \leq 1\}$$

$$\leq \sup\{\|T_1(x)\| : \|x\| \leq 1\} + \sup\{\|T_2(x)\| : \|x\| \leq 1\}$$

$$\leq \|T_1\| + \|T_2\|$$

$\therefore$ To prove $\|\alpha T\| = |\alpha| \|T\|$:

consider $\|\alpha T\| = \sup\{\|\alpha T(x)\| : \|x\| \leq 1\}$

$= \sup\{|\alpha| \|T(x)\| : \|x\| \leq 1\}$

$= |\alpha| \sup\{\|T(x)\| : \|x\| \leq 1\}$

$= |\alpha| \|T\|$

$\therefore \mathcal{B}(N, N')$ is a normed linear space

To prove $\mathcal{B}(N, N')$ is a Banach Space:-

Let $\{T_n\}$ be a cauchy's sequence in $\mathcal{B}(N,N')$

$\therefore$ By definition for a given $\epsilon > 0$ $\exists$ +ve integer 'n' $\ni \|T_n - T_m\| < \epsilon$ $\forall$ $n, m \geq n_0$ —(1)

Now, for each 'x' in N, we have $\{T_n(x)\}$ is a cauchy's sequence in N' where N' is complete.

$\Rightarrow T_n(x)$ cgs to some point $T(x)$ in N'

thus we have a new mapping $T : N \rightarrow N'$

Now we prove that the mapping 'T' is a linear transformation

For any scalar α, β

consider $T(\alpha x + \beta y) = \lim_{n\to\infty} T_n(\alpha x + \beta y)$ $(\because T(x) = \lim_{n\to\infty} T_n(x))$

$= \lim_{n\to\infty} (\alpha T_n(x) + \beta T_n(y))$

$= \alpha \lim_{n\to\infty} T_n(x) + \beta \lim_{n\to\infty} T_n(y)$

$= \alpha T(x) + \beta T(y)$

$\therefore T$ is a linear transformation.

Now we prove that T is bounded:-

Since $\{\|T_n\|\}$ is a cauchy sequence of real numbers and we know that the real number system is complete.

$\Rightarrow \|T_n\|$ converges and hence bounded.

ie, its infimum and supremum exists.

Now consider $\|T(x)\| = \|\lim_{n\to\infty} T_n(x)\|$

$= \lim_{n\to\infty} \|T_n(x)\|$

$\leq \sup \|T_n(x)\|$

$\leq \sup\{\|T_n\| \|x\|\}$ $(\because \sup \|T_n\|$ real number let it be 'k')

$\leq k \|x\|$

$\therefore T \in \mathcal{B}(N, N')$

Now it remains to prove that our cauchy's sequence $\{T_n\}$ cgs to

(4)

'T' with respect to the norm in $\mathcal{B}(N,N')$

Since $\{T_n\}$ is a cauchy sequence in $\mathcal{B}(N,N')$ and by eq^n ① we have $\|T_n - T_m\| < \epsilon$.

Now consider $\|T_n(x) - T_m(x)\| = \|(T_n - T_m)(x)\|$

$$\leq \|T_n - T_m\| \|x\|$$

$$\leq \|T_n - T_m\| \quad (\because \|x\| \leq 1)$$

$$< \epsilon$$

$\therefore \|T_n(x) - T_m(x)\| < \epsilon$ $\quad (\because$ By eq^n ①$)$

Fixing 'n' and letting $m \to \infty$, we get $\|T_n(x) - T_\infty(x)\| < \epsilon$

$$\Rightarrow \|(T_n - T)(x)\| < \epsilon$$

By taking 'ϵ' is arbitrary large, we get $(\sup\{\|(T_n - T)(x)\| : \|x\| \leq 1\} < \epsilon$

$\Rightarrow \|T_n - T\| < \epsilon)$.

and hence $T_n \to T$

$\therefore \mathcal{B}(N,N')$ is a complete normed linear space and hence Banach space.

Hence proved.

Problems:-

1. $\mathcal{B}(N)$ is the linear operator and algebra where the multiplication is defined by $\|TT'\| \leq \|T\| \|T'\|$

Consider $\|TT'\| = \sup\{\|TT'(x)\| : \|x\| \leq 1\}$

$$= \sup\{\|T(T'(x))\| : \|x\| \leq 1\}$$

$$\leq \sup\{\|T\| \|T'(x)\| : \|x\| \leq 1\}$$

$$= \|T\| \sup\{\|T'(x)\| : \|x\| \leq 1\}$$

$$= \|T\| \|T'\|$$

$\therefore \|TT'\| \leq \|T\| \|T'\|$

2. Addition and scalar multiplication are jointly continuous in $\mathcal{B}(N)$

Now we have to prove that vector addition is continuous:-

Let $\{T_n\}$ and $\{T_n'\}$ be two convergent sequences

By def for a given $\epsilon > 0$ $\exists$ a +ve integer 'n' $\ni \|T_n - T\| \leq \epsilon \; \forall n \geq N$,

$\|T_n' - T'\| \leq \epsilon \; \forall n \geq N$

Now we prove that $T_n + T_n' \to T + T'$

Now consider $\|(T_n + T_n') - (T + T')\| = \|(T_n - T) + (T_n' - T')\|$

$= \sup\{\|(T_n - T)(x) + (T_n' - T')(x)\| : \|x\| \leq 1\}$ $= \sup\{\|T_n - T + T_n' - T'\|(x) : \|x\| \leq 1\}$ $\leq \|T_n - T\| + \|T_n' - T'\|$

$= \sup\{\|T_n - T\|(x)\| : \|x\| \le 1\} + \sup\{\|T_n' - T'\|(x)\| : \|x\| \le 1\}$

$\le \|T_n - T\| + \|T_n' - T'\|$

$< \epsilon + \epsilon$

$< \epsilon \to 0$ as $n \to \infty$

Now we prove that scalar multiplication is continuous:-

Let $\{\alpha_n\}$ and $\{T_n\}$ be cgt sequencies

Now we prove that $\alpha_n T_n \to \alpha T$

Now consider $\|\alpha_n T_n - \alpha T\| = \sup\{\|\alpha_n T_n - \alpha_n T + \alpha_n T - \alpha T\|$

$= \sup\{\|\alpha_n T_n - \alpha_n T + \alpha_n T - \alpha T(x)\| : \|x\| \le 1\}$

$= \sup\{\|\alpha_n (T_n - T)(x) + T(\alpha_n - \alpha)(x)\| : \|x\| \le 1\}$

$\le \sup\{\|\alpha_n (T_n - T)(x)\| : \|x\| \le 1\} + \sup\{T(\alpha_n - \alpha)(x) : \|x\| \le 1\}$

$= |\alpha_n| \|T_n - T\| + \|T\| |\alpha_n - \alpha|$

$\to 0$ as $n \to \infty$

$\therefore \alpha_n T_n \to \alpha T$

3, prove that multiplication is jointly continuous in $B(N)$

Let $\{T_n\}$ and $\{T_n'\}$ be two cgt sequencies in $B(N)$ such that

$T_n \to T$ and $T_n' \to T'$

Now we prove that T_n and T_n' cgs to TT' :-

consider $\|T_n T_n' - TT'\| = \sup\{\|T_n T_n' - TT'(x)\| : \|x\| \le 1\}$

$= \sup\{\|T_n T_n' - T_n T' + T_n T' - TT'(x)\| : \|x\| \le 1\}$

$= \sup\{\|T_n (T_n' - T') + T'(T_n - T)(x)\| : \|x\| \le 1\}$

$\le \sup\{\|T_n (T_n' - T')(x)\| : \|x\| \le 1\} + \sup\{\|T'(T_n - T)(x)\| : \|x\| \le 1\}$

$\le \sup\{\|T_n\| \|T_n' - T'(x)\| : \|x\| \le 1\} + \sup\{\|T'\| \|T_n - T(x)\| : \|x\| \le 1\}$

$= \|T_n\| \|T_n' - T'\| + \|T'\| \|T_n - T\|$

$\to 0$ as $n \to \infty$

$\|T_n T_n' - TT'\| \to 0$ as $n \to \infty$

$T_n T_n' \to TT'$ Hence Proved

Remark:- If $N \neq \{0\}$ and I is the identity linear transformation then I is an identity in the algebra $B(N)$ such that $\|I\| = 1$

Proof:- Since 'I' is the identity linear transformation. clearly, I is an identity in $B(N)$

Now consider $\|I\| = \sup\{\|I(x)\| : \|x\| \le 1\}$

$= \sup\{\|x\| : \|x\| \le 1\}$

Definition:- Let N and N' be two normed linear spaces and 'T' be a continuous linear transformation of N into N' if T a one to one linear transformation $\ni \|T(x)\| = \|x\|$ then T is said to be an isomorphic isomorphism of N into N' (or) N is isomorphically isomorphic to N' if $\exists$ a isomorphic isomorphism between N and N'

Sec-1.8 Hahn Banach theorem

Definition:- The set $\mathcal{B}(N, \mathbb{R})$ (or) $\mathcal{B}(N, \mathbb{C})$ is the set of all continuous linear transformations from N into $\mathbb{R}$ (or) $\mathbb{C}$ and this space is called as "Conjugate space". It is denoted by N^*

Note:- 1. The elements in N^* are called "functionals" or "linear functionals".

2. The norm of a functional is given by $\|f\| = \sup\{|f(x)| : \|x\| \ldots$ or $\inf\{k : k > 0 \text{ \& } |f(x)| \leq k\|x\|\}$

3. N^* is a normed linear space with respect to the norm defined above.

******

Lemma:- Let 'N' be a normed linear space and 'M' be a closed linear subspace of 'N'. If x_0 is a vector not in M and 'f' is the functional defined on M. If M_0 is the linear space spanned by 'M' and x_0 then 'f' can be extended to a functional f_0 such that $\|f_0\| = \|f\|$.

Given that 'N' be a normed linear space and 'M' be a closed linear subspace of 'N'

If x_0 is a vector not in M and 'f' is the functional defined on M

Also M_0 is the linear space spanned by M and x_0

claim:- 'f' can be extended to a functional f_0 on $M_0 \ni \|f_0\| \ldots$

We prove this lemma in two cases.

ie, when 'N' is taken over the real field and N is taken over the complex field.

case I :- when 'N' is taken over the real field 'R'.

Since 'f' is the functional defined on M.

without loss of generality, assume that $\|f\| = 1$

Since x_0 is the vector not in M and $M_0 = M + [x_0]$

$\Rightarrow$ every vector y in M_0 can be uniquely expressed as $y = x + \alpha x_0$
for some $\alpha \in R$, $x \in M$.
It is a unique representation, because
if $y = x_1 + \alpha x_0$, $y = x_2 + \beta x_0$
$\Rightarrow x_1 + \alpha x_0 = \beta x_0 + x_2$
$\Rightarrow (x_1 - x_2) = (\beta - \alpha) x_0$
$\Rightarrow (\beta - \alpha) x_0 \in M$ ($\because x_1 - x_2 \in M$)
$\Rightarrow x_0 \in M$
which is a contradiction to our hypothesis.
Hence, the representation of 'y' is unique.
Now define 'f_0' as $f_0 y = f_0(x + \alpha x_0)$

$$= f_0 x + f_0 \alpha x_0$$

$$= f_0(x) + \alpha r_0 \quad (\because r_0 = f_0 x_0)(f/m = f)$$

$$\therefore f_0 y = f_0(x) + \alpha r_0 \text{ --- (1)}$$

To prove f_0 is a linear transformation:-
Let β, γ be any two real numbers
consider $f_0(\beta y_1 + \gamma y_2) = f_0(\beta(x_1 + \alpha x_0)) + \gamma(x_2 + \beta x_0))$

$$= f_0(\beta x_1 + \alpha \beta x_0 + \gamma x_2 + \beta \gamma x_0)$$

$$= f_0(\beta x_1 + \gamma x_2 + (\beta + \gamma)\alpha x_0)$$

$$= f_0(\beta x_1 + \gamma x_2) + (\beta + \gamma) f_0(\alpha x_0)$$

$$= \beta f_0(x_1) + \gamma f_0(x_2) + (\beta + \gamma)\alpha r_0$$

$$= \beta f_0(x_1) + \gamma f_0(x_2) + (\beta + \gamma)\alpha r_0$$

$$= \beta(f_0(x_1) + \alpha r_0) + \gamma(f_0(x_2) + \alpha r_0)$$

$\therefore f_0(\beta y_1 + \gamma y_2) = \beta f_0(y_1) + \gamma f_0(y_2)$
Thus 'f' can be extended linearly to M_0 for every of the number
$r_0 = f_0(x_0)$
Now we have to prove that $\|f_0\| \le 1$:-
For this we have to prove that $\|f_0(x + \alpha x_0)\| \le \|x + \alpha x_0\|$ --- (2)
$[\because \|f_0(x + \alpha x_0)\| \le \|f_0\| \, \|x + \alpha x_0\|$ and
if $\|f_0\| = 1$ we have $\|f_0(x + \alpha x_0)\| \le \|x + \alpha x_0\|]$
Now by eq^n (1) we have $f_0(x + \alpha x_0) = f(x) + \alpha r_0$
substitute this value in eq^n (2)
$\|f(x + \alpha x_0)\| \le \|x\| + \|x + \alpha x_0\|$ ~~$\le \|x + \alpha x_0\|$~~
$\Rightarrow$ ~~$\le f(x) + \alpha r_0$~~ $- \|x + \alpha x_0\| \le f(x) + \alpha r_0 \le \|x + \alpha x_0\|$

$\Rightarrow -f(x) - \|x+\alpha x_0\| \leq t\alpha x_0 \leq -f(x) + \|x+\alpha x_0\|$

$\Rightarrow -\frac{f(x)}{\alpha} - \|\frac{x}{\alpha} + x_0\| \leq x_0 \leq \|\frac{x}{\alpha} + x_0\| - f(\frac{x}{\alpha})\|$ —(3)

Now for any two vectors x_1, x_2 in M we have $f(x_2) - f(x_1) = f(x_2 - x_1)$

$$|f(x_2) - f(x_1)| \leq |f(x_2 - x_1)|$$
$$\leq \|f\| \|x_2 - x_1\|$$
$$\leq \|x_2 - x_1\|$$
$$\leq \|x_2 - x_0 + x_0 - x_1\|$$
$$\leq \|x_2 + x_0\| + \|x_1 + x_0\|$$
$$\therefore -f(x_1) - \|x_1 + x_0\| \leq \|x_2 + x_0\| - f(x_2) \text{ —(4)}$$

Let $a = \sup\{-f(x) - \|x + x_0\| : x \in M\}$

and $b = \inf\{-f(x) + \|x + x_0\| : x \in M\}$

Then by eq^n (4) we have $a \leq b$ and by eq^n (3) we have $a \leq x_0 \leq b$

Thus if we choose x_0 in such a way that $a \leq x_0 \leq b$ then our claim is over

ie, f_0 is the functional and $\|f_0\| = 1$.

Case 1:- when 'N' is taken over the complex field:-

Let 'f' be a complex functional defined on $M \ni \|f\| = 1$

ie, $f(x) = g(x) + i h(x)$ —(5) where g & h are real-valued functionals defined on 'M'

Since $\|g\| \leq \|f\|$ and $\|f\| = 1$

$\Rightarrow \|g\| = 1$

Now consider $f(ix) = g(ix) + i(h(ix))$ and $if(x) = ig(x) - h(x)$

But we have $f(ix) = if(x)$

$\therefore$ By comparing the real parts, we have $h(x) = -g(ix)$

$\therefore$ Substitute this value in eq^n (5) we have $f(x) = g(x) - ig(ix)$ —(6)

Since 'g' is a real-valued functional defined on 'M' and by (...) we can extend 'g' to a real valued functional 'g_0' on M_0

such that $\|g_0\| = \|g\|$

Now for some $x \in M_0$ defined f_0 as $f_0(x) = g_0(x) - ig_0(ix)$ —(7)

(: By eq^n (6))

Then we have to prove that f_0 is a linear transformation

for any $x, y \in M_0$

consider $f_0(x+y) = g_0(x+y) - ig_0(i(x+y))$

$= g_0(x+y) - ig_0(ix) - ig_0(iy)$

$= g_0(x) + g_0(y) - i g_0(ix) - i g_0(iy)$

$= [g_0(x) - i g_0(ix)] + [g_0(y) - i g_0(iy)]$

$= f_0(x) + f_0(y)$

consider $f_0(\alpha x) = g_0(\alpha x) - i g_0(i\alpha x)$

$= \alpha g_0(x) - \alpha i g_0(ix)$

$= \alpha [g_0(x) - i g_0(ix)]$

$= \alpha [f_0(x)]$

$f_0(\alpha x) = \alpha (f_0(x))$

Hence f_0 is a linear extension of f from M to M_0 for all Real α's

If α is a complex numbers:-

Put $\alpha = i$ in the eq^n $f_0(\alpha x) = g_0(\alpha x) - i g_0(i\alpha x)$

Thus, we get $f_0(ix) = g_0(ix) - i g_0(i^2 x)$

$= g_0(ix) + i g_0(x)$

Now $i f_0(x) = i g_0(x) - i^2 g_0(ix)$

$= i g_0(x) + g_0(ix)$

But we have $f_0(ix) = i f_0(x)$

$\Rightarrow g_0(ix) + i g_0(x) = i g_0(x) + g_0(ix)$

Thus f_0 is linear as a complex valued functional defined on the complex space M_0

Now it remains to prove that $\|f_0\| = 1$ as $\|f\| = 1$

Let $x \in M_0$ and $\|x\| = 1$

Now we prove that $|f_0(x)| \leq 1$

If $f_0(x)$ is real then $f_0(x) = g_0(x)$ and $\|g_0\| = \|g\| = 1$

$\Rightarrow \|f_0\| = 1$

If f_0 is a complex then we can write $f_0(x) = r e^{i\theta}$ for some $r > 0$

consider $|f_0(x)| = |f_0\, r e^{i\theta}| = r(1)$ $(|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1)$

$= r$

$= e^{-i\theta} r e^{i\theta}$

$= e^{-i\theta} f_0(x)$

$= f_0\, e^{-i\theta} x$

But $\|e^{-i\theta}(x)\| = |e^{-i\theta}| \|x\|$

$= |e^{-i\theta}(1)| = 1$

$\therefore |f_0(x)| = |f_0\, e^{-i\theta}(x)| \leq \|e^{-i\theta}(x)\| \|f_0\|$

$= (1) \|f_0\|$

which is less than ≤ 1

or equal to 1. Hence $\|f_0\| = 1 = \|f\|$

Theorem: A the Hahn Banach theorem

Statement: Let 'M' be a linear Subspace of a normed linear Space 'N' and 'f' be a functional defined on 'M' then 'f' be extended to a functional f_0 defined on the whole space 'N' $\ni$ $\|f_0\| = \|f\|$

Proof: Let 'S' be the set of all extensions 'g' of 'f' with the same norm defined on subspaces which contain 'M'

Then 'S' is a poset with respect to the following relation $g_1 \leq g_2$ means that the domain of g_1 contained in the domain of g_2 and $g_2(x) = g_1(x)$ $\forall$ x in the domain of g_1

Then clearly $(g, \leq)$ is a poset.

Let 'p' be the chain of the poset 'S'

$\because$ The union of any chain of extensions is also an extension and is therefore, an upper bound for the chain.

$\therefore$ By Zorn's lemma, $\exists$ a maximal extension f_0 and the domain of f_0 must be 'N'.

for, otherwise it could be extended further more by above lemma which is a contradiction to the maximality of f_0

Thus 'f' can be extended to a functional f_0 on the entire space 'N' $\ni$ $\|f_0\| = \|f\|$ (By above lemma)

Hence proved.

Applications of Hahn-Banach Theorem:-

Theorem: B Let 'N' be a normed linear space and 'x_0' is a non zero vector in N then $\exists$ a functional f_0 in N^* $\ni$ $f_0(x_0) = \|x_0\|$ and $\|f_0\| = 1$

Proof:- Given that 'N' is the normed linear space and x_0 is a non zero vector in 'N'.

Claim:- $\exists$ a functional f_0 in N^* $\ni$ $f_0(x_0) = \|x_0\|$ and $\|f_0\| = 1$

Let $M = \{\alpha x_0\}$ is the linear subspace of N spanned by x_0

Defined 'f' on M by $f_0(\alpha x_0) = \alpha \|x_0\|$

For this, first we prove that 'f' is a linear transformation.

Let $y_1, y_2 \in M$ $\ni$ $y_1 = \alpha x_0$, $y_2 = \beta x_0$

Let γ, δ be any scalars.

consider, $f(\gamma y_1 + \delta y_2) = f(\gamma(\alpha x_0) + \beta(x_0)\delta)$

$= f((\gamma\alpha + \beta\delta)x_0)$

$= (\gamma\alpha + \beta\delta)\|x_0\|$

$= \gamma\alpha\|x_0\| + \delta\beta\|x_0\|$

$= \gamma f(\alpha x_0) + \delta f(\beta x_0)$

$f(\gamma y_1 + \delta y_2) = \gamma f(y_1) + \delta f(y_2)$

$\therefore$ 'f' is a linear transformation.

In order to prove that 'f' is continuous we have to prove that 'f' is bounded.

For this consider $|f(y)| = |f(\alpha x_0)|$

$= |\alpha|\,\|x_0\|$

$\therefore |f(y)| = |\alpha|\,\|x_0\|$ — (1)

Since $\|y_0\| = \|\alpha x_0\|$

$= |\alpha|\,\|x_0\|$ — (2)

from (1) and (2) $|f(y)| = \|y_0\|$

$= 1 \cdot \|y\|$

$\leq a \cdot \|y\|$

and hence 'f' is bounded.

Thus, f is a functional defined on M also we have $\|f\| = \sup\{|f(y)| : y \in M\}$ and $\|y\| \leq 1$

$= \sup\{\|y\| : y \in M\}$ and $\|y\| \leq 1$

$= 1$

Since $f(\alpha x_0) = \alpha\|x_0\|$

$\Rightarrow f(x_0) = \|x_0\|$

where 'f' is the functional defined on 'M'.

By hahn-banach we have theorem 'f' can be extended to a functional f_0 such that $f_0(x_0) = \|x_0\|$ and $\|f_0\| = 1$

Hence proved.

Theorem:- If 'M' is a closed linear subspace of a normed linear space 'N' and x_0 is a non-zero vector in 'M' then $\exists$ a functional f_0 in N^* $\ni$ $f_0(M) = 0$ and $f_0(x_0) \neq 0$

Proof:- consider the quotient space $\frac{N}{M}$

Define $T: N \to \frac{N}{M}$ by $T(x) = x + M \;\forall\; x \in N$

Now we have to prove that 'T' is a continuous linear transformation.

Let $x, y \in N$ and α, β be any scalars.

Consider $T(\alpha x + \beta y) = (\alpha x + \beta y) + M$

$= (\alpha x + M) + (\beta y + M)$

$= \alpha(x+M) + \beta(y+M)$

$= \alpha T(x) + \beta T(y)$

$\therefore T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$

To prove 'T' is continuous:-

If $\{x_n\} \to 0$

$\Rightarrow T(x_n) \to 0 + M$

ie, $T(x_n) \to 0$

and since 'M' is a closed linear subspace of 'N' it contains all its limit points.

$\therefore (T(x_n)) \to 0$ if $x_n \to 0$

Hence 'T' is continuous.

for some $m \in M$, we have $T(m) = \underset{=0}{m+M}$

Also if x_0 is a non-zero vector in 'N' then $T(x_0) = x_0 + M \neq 0$

Then by applying theorem-B,

$\exists$ a functional 'f' in $\left(\frac{N}{M}\right)^* \ni f(x_0 + M) = \|x_0 + M\| = 1$

Now define f_0 as $f_0(x) = f(T(x))$

Clearly $f_0 \in N^*$

Now we prove that 'f_0' is a continuous linear transformation:-

for any scalars α, β

Consider $f_0(\alpha x + \beta y) = f(T(\alpha x + \beta y))$

$= f(\alpha T(x) + \beta T(y))$

$= \alpha f(T)(x) + \beta f(T(y))$

$= \alpha f_0(x) + \beta f_0(y)$

$\therefore f_0(\alpha x + \beta y) = \alpha f_0(x) + \beta f_0(y)$

In order to prove that f_0 is continuous we prove that 'f_0' is bounded.

consider $|f_0(x)| = |f(T(x))|$

$\leq \|f\| \|T(x)\|$

$\leq \|f\| \|T\| \|x\|$

Since 'f' is bounded.

$\Rightarrow f_0$ is also bounded.

$\therefore f_0$ is a continuous linear transformation.

Also if $m \in M$, $f_0(m) = f(T(m))$

$= f(0)$

$= 0$

$\therefore f_0(n)=0$

If 'x_0' is a vector not in 'M' then $f_0(x_0) = f(\tau(x_0))$

$= f(x_0+M)$

$\neq 0$

$\therefore f_0(x_0) \neq 0$

$\therefore f_0$ satisfies the desired properties.

Hence proved.

The Natural Embedding of N in N^{**} ←

Definition:- Let 'N' be a normed linear space. The space $(N^*)^*$ $=N^{**}$ is called the second conjugate space of 'N' (or) the conjugate space of N^* where the elements in N^{**} are functionals from N^* to $\mathbb{R}$ or $\mathbb{C}$.

Theorem:- Let 'N' be a normed linear space then every vector 'x' in N induces some functional F_x in N^{**} given by $F_x(f) = f(x)$ & $\|F_x\| = \|x\|$ further the mapping $x \to F_x$ is an isomorphic isomorphism.

Proof:- Given that 'N' be a normed linear space and $x \in N$

claim:- 1, x induces some functional F_x in N^{**} such that $\|F_x\|=\|x\|$

2, the mapping x to F_x is an isomorphic isomorphism.

To prove 1:- In order to prove that F_x is a functional.

First we prove that F_x is a linear functional.

Let $f, g \in N^*$ and α, β be any scalars.

consider $F_x(\alpha f+\beta g) = \alpha f+\beta g(x)$

$= \alpha f(x)+\beta g(x)$

$= \alpha F_x(f)+\beta F_x(g)$

$\therefore F_x$ is a linear transformation.

Now we prove that F_x is bounded.

Consider $|F_x(f)| = |f(x)|$

$\leq \|f\| \|x\|$

$\therefore F_x$ is bounded

Since 'f' is a functional.

$\therefore F_x$ is bounded and hence continuous.

Now we prove that $\|F_x\| = \|x\|$:-

By definition of norm of a functional, we have $\|F_x\| = \sup\{|F_x(f)| : \|f\| \leq 1\}$

$= \sup\{|f(x)| : \|x\| \leq 1\}$ $\|(\ldots - x)\| <$

$\leq \sup\{\|f\|\|x\| : \|f\| \leq 1\}$

$\leq \|x\| \sup\{\|f\| : \|f\| \leq 1\}$

$\leq \|x\| \cdot (1)$

$\leq \|x\|$

$\therefore \|F_x\| \leq \|x\|$ —— (1)

In order to prove the reverse inequality we have two cases

Case I: If $x = 0$

$\Rightarrow \|F_0\| = \|0\| = 0$

But since norm is always non-ve, $\|F_0\| \geq 0$

$\therefore \|F_0\| = 0$

$\Rightarrow \|F_x\| = \|x\|$

Case II: If $x \neq 0$

Since 'x' is a non-zero vector in 'N'

By theorem-B $\exists$ a functional 'f_0' in N^* $\ni$ ($|f(x)| = \|x\|$ and $\|f\| = 1$ —— (2)

Since $\|F_x\| = \sup\{|F_x(f)| : \|f\| \leq 1\}$

$\|F_x\| = \sup\{|f(x)| : \|f\| \leq 1\}$ —— (3)

By eqn (2) we have $\|x\| = |f(x)| \leq \sup\{|f(x)| : \|f\| \leq 1\}$

$\Rightarrow \|x\| = |f(x)| \leq \|F_x\|$

$\Rightarrow \|x\| \leq \|F_x\|$ —— (4)

From (1) and (4) we have $\|x\| = \|F_x\|$

Thus,

The mapping $x \to F_x$ is a norm-preserving mapping.

To prove I:- To prove this first we prove that $x \to F_x$ is a lin-ear transformation.

consider $F_{x+y}(f) = f(x+y) = f(x) + f(y)$.

$= F_x(f) + F_y(f)$

consider $F_{\alpha x}(f) = f(\alpha x) = \alpha f(x)$

$= \alpha F_x(f)$

$\therefore x \to F_x$ is a linear transformation.

To prove the mapping $x \to F_x$ is one-one :-

Let $F_x = F_y$

$\Rightarrow \|F_x\| = \|F_y\|$

$\Rightarrow \|F_x - F_y\| = 0$

$\Rightarrow \|(F_x - F_y)\| = 0$

$\Rightarrow \|F_x - y\| = 0$

$\Rightarrow \|x - y\| = 0$

$\Rightarrow x - y = 0 \Rightarrow x = y$

$\therefore$ The mapping $X \to F_x$ is one-one.

$\therefore$ The mapping $X \to F_x$ is, isomorphic isomorphism.

Hence proved.

Note: 1, The mapping $x \to f_x$ is a norm-preserving mapping of N into N^{**} and f_x is called the functional on N^* induced by the vector x in N. (or) simply called "Induced-functional".

2, The mapping $x \to f_x$ is an isomorphic isomorphism of N to N^{**} and is called as "Natural embedding of N in N^{**}."

(prove it we write above Theorem)

Definition:- The weak Topology on N is defined to be the weak topology generated by the elements in N^*. ie, It is the weakest topology on N with respect to which all the functionals in N^* are continuous.

Weak* Topology:- The weak* topology on N^* is defined to be the weak topology on N^* generated by all the induced functionals f_x in N^{**}. ie, It is the weakest topology on N^* with respect to which all the elements in N^{**} are continuous.

Definition:- Let x be a vector in N and f_0 is the arbitrary functional on N^* and let $\epsilon > 0$ be given then the neighbourhood of f_0 is given by $S(x, f_0, \epsilon) = \{f / f \in N^* \text{ and } |F_x(f) - F_x(f_0)| < \epsilon\}$
$= \{f / f \in N^* \text{ and } |f(x) - f_0(x)| < \epsilon\}$

where the set $S(x, f_0, \epsilon)$ is called "the neighbourhood of f_0 in the weak* topology on N^*".

Note:- 1, The class of these sets (neighbourhood's) forms an open subbase for the weak* topology on N^*.

2, All finite intersection of these sets constitute an open base for the weak* topology on N^*.

3, N^* is a housdroff space with respect to its weak* topology.

4, Consider the sphere $S^* = \{f / f \in N^*\}$ and $\|f\| \leq 1\}$ is a closed unit sphere in N^*.

5, S^* is compact in the strong topology $\Leftrightarrow$ N is finite dimensional.

8, S^* is compact in the weak topology $\Leftrightarrow$ N is Reflexive

Definition:- A normed linear space 'N' is reflexive if $N = N^{**}$

Note:- Always $N \subseteq N^{**}$

Theorem:- 8 If 'N' is a normed linear space then the closed unit sphere S^* in N^* is a compact housdroff space in the weak* topology.

Proof:- Given that 'N' is a normed linear space $S^* = \{f \mid f \in N^*$ and $\|f\| \leq 1\}$ is the closed unit sphere in N^*

Claim:- S^* is a compact housdroff space in the weak* topology.

To prove S^* is a housdroff space.

Let f and g are two disjoint points in S^*

ie, $f \neq g$

we have to prove that $\exists$ two disjoint neighbourhoods $S(x, f, \epsilon)$ and $S(x, g, \epsilon)$ for 'f' and 'g' respectively.

On the contrary way, suppose that $S(x, f, \epsilon) = S(x, g, \epsilon)$

where $S(x, f, \epsilon) = \{f_0 \mid f_0 \in N^*$ and $|f(x) - f_0(x)| < \epsilon\}$

$S(x, g, \epsilon) = \{g_0 \mid g_0 \in N^*$ and $|g(x) - g_0(x)| < \epsilon\}$

Consider $|f(x) - g(x)| = |f(x) - f_0(x) + f_0(x) + g_0(x) - g_0(x) - g(x)|$

$\leq |f(x) - f_0(x)| + |f_0(x) - g_0(x)| + |g_0(x) - g(x)|$

$\leq \epsilon/3 + \epsilon/3 + \epsilon/3$

$\leq \epsilon$

As 'ϵ' is arbitrary we have $f(x) = g(x)$

which is a contradiction to our assumption.

$\therefore \exists$ two disjoint neighbourhood's and hence S^* is Housdroff Space.

Now it remains to prove that S^* is compact.

Let C_x is the closed interval $[-\|x\|, \|x\|]$ (or) the closed disc $\{z \mid |z| \leq \|x\|\}$ according N is real and complex.

Now we prove that S^* is a compact space with each vector in N.

Since C_x is compact [$\because$ as every closed interval in the real line is compact and the disc is compact]

$\Rightarrow$ By Tychnoff's theorem

ie, the product of compact spaces $C = \prod_x C_x$ is also compact

for each 'x' the values of $f(x)$ of all f's in S^* lies in C_x.

Thus, we can embedded S^* in C (by regarding each f in S^* as identical with the array of all its values at the vector x in N) from the definition we have the weak* topology on S^* equals its topology as a subspace of 'C'

But we know that 'C' is compact then it is sufficient to prove that S^* is a closed subspace of 'C'

i.e, we have to prove that $\overline{S^*} = S^*$

we know that $S^* \subseteq \overline{S^*}$

To find the reverse inclusion $\overline{S^*} \subseteq S^*$:-

If $g \in \overline{S^*}$ then $g \in S^*$

Let 'g' be a function defined on the index set N, then since $g \in C$

$\Rightarrow |g(x)| \leq \|x\| \;\forall\; x \in N$

It is sufficient to prove that 'g' is a linear function defined on 'N'.

Let $\epsilon > 0$ be given

Let 'x' & 'y' be two vectors in 'N'.

Every basis neighbourhood of 'g' intersects S^* so $\exists\, f \in S^*$ $\ni$

$|g(x) - f(x)| < \epsilon/3$

$\Rightarrow |g(x+y) - f(x+y)| < \epsilon/3$

$\Rightarrow |g(y) - f(y)| < \epsilon/3$

Since 'f' is linear

$\Rightarrow f(x+y) - f(x) - f(y) = 0$

$|g(x+y) - g(x) - g(y)| = |[g(x+y) - f(x+y)] - (g(x) - f(x)) - (g(y) - f(y))|$

$\leq |g(x+y) - f(x+y)| + |g(x) - f(x)| + |g(y) - f(y)|$

$< \epsilon/3 + \epsilon/3 + \epsilon/3$

$< \epsilon$

$\therefore |g(x+y) - g(x) - g(y)| < \epsilon$ where 'ϵ' is arbitrary constant.

$\therefore g(x+y) = g(x) + g(y)$

Similarly we prove that $g(\alpha x) = \alpha g(x)$.

$\therefore g$ is linear

Hence proved.

The Open Mapping Theorem

Lemma:- Let B and B' are two Banach spaces and 'T' is a continuous linear transformation of B onto B'. Then the image of each open sphere centered at the origin in 'B' contains an open sphere centered at the origin in B'

Proof:- Given that B and B' are two Banach spaces and 'T' is continuous linear transformation of B onto B'.

Let S_r and S_r' are two spheres centered at the origin in B and B' respectively.

Claim:- $T(S_r) \supseteq T(S_r')$

Let $S(x,r)$ is the open sphere center at 'x' with radius 'r'

Let $y \in S(x,r)$

$\Leftrightarrow \|y-x\| < r$

$\Leftrightarrow \|z\| < r$ where $z = y-x$

$\Leftrightarrow y = x+z$ and $\|z\| < r$

$\Leftrightarrow y \in x + S_r$, where $S_r = \{z / \|z\| < r\}$

$\therefore S(x,r) = x + S_r$ — (1)

Since $S_r = \{x / \|x\| < r\} = \{x / \frac{\|x\|}{r} < 1\}$

Let $\frac{x}{r} = y \Rightarrow x = yr$

$\therefore S_r = \{yr / \frac{\|x\|}{r} < 1\} \Rightarrow S_r = \{yr / \|y\| < 1\}$

$\Rightarrow S_r = r\{y / \|y\| < 1\}$

$\Rightarrow S_r = r S_1$ — (2)

Since $S_r = r S_1$

$\Rightarrow T(S_r) = T(rS_1)$

$\Rightarrow T(S_r) = rT(S_1)$ — (3)

Then it is sufficient to prove that $T(S_1) \supseteq$ some open sphere S_ϵ' for this

first we prove that $\overline{T(S_1)} \supseteq$ some open sphere S_ϵ'

For each +ve integer 'n' consider an open sphere S_n in B $\ni$

$B = \bigcup_{n=1}^{\infty} S_n$

Since 'T' is continuous linear transformation of B onto B'

$\Rightarrow B' = T(B)$

$\Rightarrow B' = T\left(\bigcup_{n=1}^{\infty} S_n\right)$

$\Rightarrow B' = \bigcup_{n=1}^{\infty}(T(S_n))$

Since B' is a complete metric space, By Baire's theorem $\exists$ some $\overline{T(S_{n_0})}$ has an interior point 'y_0'

Assume that the interior point y_0 lies in $\overline{T(S_{n_0})}$

Now define a homeomorphism $f: B \to B'$ by $f(y) = y - y_0$

we use this mapping 'f' to prove zero is the interior point of $\overline{T(S_{n_0})} - y_0$

Since 'y_0' is the interior point of $\overline{T(S_{n_0})}$

$\Rightarrow \exists$ an open set G such that $y_0 \in G \subset \overline{T(S_{n_0})}$

$\Rightarrow f(y_0) \in f(G) \subseteq f(\overline{T(S_{n_0})})$

$\Rightarrow y_0 - y_0 \in f(G) \subset (\overline{T(S_{n_0})} - y_0)$

$\Rightarrow 0 \in f(G) \subset \overline{T(S_{n_0})} - y_0$

$\therefore$ zero is the interior point of $\overline{T(S_{n_0})} - y_0$ —(3)

Now we have to prove that $T(S_{n_0}) - y_0 \subset T(S_{2n_0})$

Let $y \in T(S_{n_0}) - y_0$

then $\exists$ some $x \in S_{n_0} \ni y = T(x) - y_0$

But $y_0 \in T(S_{n_0}) \Rightarrow y_0 = T(x_0)$ for some $x_0 \in S_{n_0}$

thus $y = T(x) - T(x_0)$

$= T(x - x_0)$

$y = T(x - x_0)$ —(4)

But we have $(x, x_0) \in S_{n_0}$

$\Rightarrow \|x\| < n_0$ and $\|x_0\| < n_0$

$\Rightarrow \|x - x_0\| \leq \|x\| + \|x_0\|$

$< n_0 + n_0 < 2n_0$

$\therefore x - x_0 \in S_{2n_0}$

$\Rightarrow T(x - x_0) \in (S_{2n_0})$

$\Rightarrow y \in (S_{2n_0})$ ($\because$ eq^n (4))

$\therefore T(S_{n_0}) - y_0$ is contained in $T(S_{2n_0})$ ie, $T(S_{n_0}) \subseteq T(S_{2n_0})$

$\Rightarrow \overline{T(S_{n_0}) - y_0} \subseteq \overline{T(S_{2n_0})}$ —(5)

Since 'f' is homeomorphism $f(\overline{T(S_{n_0})}) \subseteq \overline{f(T(S_{n_0}))}$

$f(\overline{T(S_{n_0})}) - y_0 \subseteq \overline{T(S_{n_0}) - y_0}$

$\subseteq \overline{T(S_{2n_0})}$ (By eq^n (5))

$\therefore \overline{T(S_{n_0})} - y_0 \subseteq \overline{T(S_{2n_0})}$ —(6)

Thus zero is an interior point of $\overline{T(S_{2n_0})}$

Again define a homeomorphism $g : B \to B'$ by $g(x) = 2n_0 x$

Thus, $\overline{g(T(S_1))} = g(\overline{T(S_1)})$

$\Rightarrow \overline{2n_0(T(S_1))} = 2n_0\overline{T(S_1)}$ (By eq^n (6))

Now $\overline{T(S_{n_0})} - y_0 \subseteq 2n_0\overline{T(S_1)}$ — (7)

From (3) and (7), we have '0' is the interior point of $\overline{T(S_1)}$

Thus $\exists\ \epsilon > 0 \ni S'_{\epsilon} \subseteq \overline{T(S_1)}$ — (8)

We conclude the proof by showing that $S'_\epsilon \subset T(S_3)$ which is equivalent to $S'_\epsilon \subset T(S_1)$

Let 'y' be an arbitrary point on S'_ϵ

$\Rightarrow \|y\| < \epsilon$

We have to prove that $S'_\epsilon \subset T(S_3)$

Since 'y' is an arbitrary point on S'_ϵ

By eq^n (8) 'y' is an element of $\overline{T(S_1)}$

$\Rightarrow y$ is a limit point of $T(S_1)$ and hence $\exists$ a vector y_1 in $T(S_1) \ni$

$\|y - y_1\| < \epsilon/2$ But $y_1 \in T(S_1)$

$\Rightarrow$ Some $x_1 \in S_1$ such that $y_1 = T(x_1)$ and $\|x_1\| < 1 = \frac{1}{2^0}$

Again by (8) we have $S'_{\epsilon/2} \subseteq \overline{T(S_{1/2})}$

Since $\|y - y_1\| < \epsilon/2$

$\Rightarrow y - y_1 \in S'_{\epsilon/2} \subseteq \overline{T(S_{1/2})}$

as before $\exists\ y_2$ in $T(S_{1/2}) \ni \|y - y_1 - y_2\| < \epsilon/2^2$

On continuing this process we get a sequence $\{x_n\}$ in B

Such that $\|x_n\| < \frac{1}{2^{n-1}}$ and $\|y - (y_1 + y_2 + y_3 + \cdots y_n)\| < \epsilon/2^n$

$\Rightarrow \lim_{n\to\infty} (y_1 + y_2 + y_3 + \cdots + y_n) = y$ — (9)

where $y_n = T(x_n)$, $\|x_n\| < \frac{1}{2^{n-1}}$

Now if we put

$t_n = x_1 + x_2 + x_3 + \cdots + x_n$ — (a)

$\|t_n\| = \|x_1 + x_2 + x_3 + \cdots + x_n\|$

$\leq \|x_1\| + \|x_2\| + \|x_3\| + \cdots + \|x_n\|$

$< 1/2^0 + 1/2^1 + 1/2^2 + \cdots + 1/2^{n-1}$ $\quad \frac{1}{1-r} = \frac{1}{1-1/2} = 2$

$= 2 < 3$

$\therefore \|t_n\| < 3$ — (10)

Also for $n > m$ we have

$\|t_n - t_m\| = \|x_{m+1} + x_{m+2} + \cdots + x_n\|$

$\leq \|x_{m+1}\| + \|x_{m+2}\| + \cdots + \|x_n\|$

$< \frac{1}{2^m} + \frac{1}{2^{m+1}} + \cdots + \frac{1}{2^{n-1}}$

$\to 0$ as $m, n \to \infty$

Hence $\{t_n\}$ is a cauchy sequence in B.

Since B is complete, $\exists$ a vector in B $\ni t_n \to x$

ie, $\lim_{n\to\infty} t_n = x$

consider $\|x\| = \|\lim_{n\to\infty} t_n\|$

$= \lim_{n\to\infty} \|t_n\|$

< 3 ($\because$ by ⑩)

Since $\|x\| < 3$ we have $x \in S_3$

Now consider, $y_1 + y_2 + y_3 + \cdots + y_n = T(x_1) + T(x_2) + \cdots + T(x_n)$

$= T(x_1 + x_2 + x_3 + \cdots + x_n)$

$= T(t_n)$ —— (11) ($\because T$ is linear transformation)

Since T is continuous and $x = \lim_{n\to\infty} t_n$

$T(x) = T(\lim_{n\to\infty} t_n)$

$= \lim_{n\to\infty} T(t_n)$

$= \lim_{n\to\infty} (y_1 + y_2 + y_3 + \cdots + y_n)$ ($\because$ by ⑪)

$T(x) = y$ (by ⑨)

Thus, $T(x) = y$ where $\|x\| < 3$

so that $y \in T(S_3)$ and we have taken $y \in S_\epsilon'$ and we get

$S_\epsilon' \subseteq T(S_3)$

Hence proved.

Open Mapping theorem:- If B and B' are two banach spaces and T is a continuous linear transformation of B' onto B then T is an "open mapping".

Proof:- Given that B and B' are two banach spaces and T is a continuous linear transformation of B onto B'.

Claim:- T is an open mapping.

ie, we have to prove that $T(G)$ is open in B' whenever G is open in B.

Let $y \in T(G) \Rightarrow y = T(x)$ for some $x \in G$

Since G is open in B & $x \in G$

$\Rightarrow \exists$ an open sphere $S(x, r)$ in B centered at 'x' with radius 'r' $\ni S(x,r) \subseteq G$ —(1)

But $S(x, r) = x + S_r \subseteq G$

By above lemma $\exists$ an open sphere S_ϵ' in B' centered at origin $\ni T(S_r) \supseteq S_\epsilon'$

ie, $S_\epsilon' \subseteq T(S_r)$

$\therefore y + S_\epsilon' \subseteq y + T(S_r)$

$= T(x) + T(S_r)$

$= T(x + S_r)$

$= T(S(x, r)) \subseteq T(G)$

$\therefore y + S_\epsilon' \subseteq T(G)$ or $S'(y, \epsilon) \subseteq T(G)$

ie, 'y' is an interior point of $T(G)$

$\Rightarrow T(G)$ is open in B'.

By definition, T is an open mapping.

Hence proved.

Theorem:- A one to one continuous linear transformation of one banach space onto another banach space then is a homomorphism. In particular if a one to one linear transformation 'T' of banach space onto itself is continuous then its inverse T^{-1} is automatically continuous.

Proof:- Given that B and B' are two banach spaces and T: B→B' is a one to one continuous linear transformation.

In order to prove that T is continuous homomorphism.

i, T is continuous.

ii, T is bijection.

iii, T^{-1} is continuous.

But already we have, given that T is conti. & bijection.

In order to prove T^{-1} is continuous. It is enough to prove T is open mapping.

By open mapping theorem, T is open mapping.

By open mapping theorem, T is open mapping.

$\Rightarrow T^{-1}$ is continuous and hence T is homomorphism.

In particular, if $T: B \rightarrow B$ is a one to one continuous linear transformation of B onto B. itself. Obviously, its inverse is also continuous.

Projection mapping:- A projection 'E' on a linear space 'L' is simply an idempotent linear transformation of 'L into itself.

i.e, $E^2 = E$

Note:- A projection 'E' on 'L' can be expressed geometrically as follows.

i, A projection E determines a pair of linear subspaces M and N ∋ $L = M \oplus N$ where $M = \{E(x) / x \in L\}$ and $N = \{x \in L / E(x) = 0\}$ are the range space and null space of E respectively.

ii, A pair of linear subspaces 'M' and 'N' ∋ $L = M \oplus N$ determines a projection E whose range spaces and null spaces are M and N respectively.

iii, If $z = \{x+y\}$ is the unique representation of a vector in 'L' as a sum of vectors in M and N. Then E is defined by $E(z) = x$

Definition:- A projection on a banach space 'B' is an idempotent operator on 'B' i.e, it is a projection operator on 'B'

i, In the algebraic sense which is also continuous.

→ **Theorem:** If 'p' is a projection on a banach space 'B' and M & N are the range and null space respectively then M & N closed linear subspaces of B & $B = M \oplus N$

Proof: Given that 'p' is a projection on B. M & N are range space and null space of 'p' respectively.

claim: M and N are closed linear subspaces of B

Since 'p' is a projection, by the property of B we have $B = M \oplus N$ where M & N are range space & null space respectively.

Now, it is enough to prove that M & N are closed linear subspaces of B.

Now by def of null space we have $N = \{x \in B \mid P(x) = 0\}$

$N = \{x \in B \mid x = P^{-1}(0)\}$

$= P^{-1}\{0\}$

∵ P is cont; & $\{0\}$ is closed in B

⇒ $P^{-1}\{0\}$ is closed in B

Thus, N is a closed subspace of B.

Now by the def of range space we have $M = \{x \in B \mid x = P(x)\}$

$= \{x \in B \mid I(x) = P(x)\}$

$= \{x \in B \mid (I - P)(x) = 0\}$

Thus, M is the null space $I - P$

∵ The identity map I is cont, and the projection 'p' is cont

⇒ $I - P$ is cont; & M is the null space of $I - P$

Thus M is also closed.

Hence proved.

Theorem: Let 'B' be a complete normed linear space and let M, N are closed linear subspaces of B ∋ $B = M \oplus N$. If $z = x + y$ is the unique representation of a vector in B as a sum of vectors in N and M then the mapping 'p' defined by $P(z) = x$ is a projection on B whose range & null spaces are M & N respectively.

Proof: Given that 'B' is a complete normed linear space and let M & N are closed linear subspaces of B ∋ $B = M \oplus N$.

If $z = x + y$ is the unique representation of a vector in B as a sum of vectors in N and M.

claim: The mapping 'p' defined by $P(z) = x$ is a projection

B whose range & null spaces are 'M' & 'N' respectively.

From the def. of a projection and by conditions (F) & (G)

it is enough to prove that 'p' is continuous.

Let B' denote the linear space B equipped with the norm

defined by $\|z'\| = \|x\| + \|y\|$

Then clearly B' is a banach space

further, $\|p(z)\| = \|x\|$

$\leq \|x\| + \|y\|$

$\leq \|z'\|$

$\therefore \|p(z)\| \leq \|z'\|$ hence 'p' is bounded.

And, therefore continuous as a mapping of B' onto B.

Now, it is sufficient to prove that B' and B have the same

topologies.

Let T denotes the identity mapping from B' onto B then

$\|T(z)\| = \|z\|$

$= \|x+y\|$

$\leq \|x\| + \|y\|$

$\leq \|z'\|$

$\therefore \|T(z)\| \leq (1)\|z'\|$

$\Rightarrow T$ is bounded & hence continuous linear transformation from

B' onto B

Then, by known theorem, T is homomorphism.

$\Rightarrow B'$ and B have the same topologies and hence 'p' is a conti.

linear transformation on B. Hence proved.

Definition: The graph of 'T' is defined as the set of all ordered pairs of the form $(x, T(x))$ where $x \in B$ & $T(x) \in B'$, B and B' are banach spaces. And also 'T' is a linear transformation of 'B' onto B' and is denoted by T_G.

ie, $T_G = \{(x, T(x)) \mid x \in B, T(x) \in B'\}$

Note:- $T_G \subseteq B \times B'$

Definition:- Let N, N' be two normed linear spaces and 'D' is a sub space of N. Then the linear transformation $T: D \xrightarrow{into} N'$ is said to be closed $\Leftrightarrow x_n \to x$ and $T(x_n) \to y$, where $x_n \in D \Rightarrow x \in D$ & $T(x) = y$!

Theorem:- Closed Graph theorem:-

If B and B' are banach space and T is a linear transformation of B into B' then T is continuous $\Leftrightarrow$ its graph is closed.

Proof:- Given that B and B' are banach space, & T is a linear continuous linear transformation of B into B'

Claim:- T is continuous $\Leftrightarrow$ Its graph is closed.

Part:1

First we suppose that T is continuous.

Let T_G be the graph of T

In order to prove that T_G is closed we have to prove that $\bar{T}_G$

But we have $T_G \subseteq \bar{T}_G$ —(1)

It remains to prove that $\bar{T}_G \subseteq T_G$

Let $(x,y) \in \bar{T}_G$

Since $\bar{T}_G = T_G \cup T_G'$

we have two cases

case i, If $(x,y) \in T_G \cup T_G' \Rightarrow (x,y) \in T_G$

$\therefore \bar{T}_G \subseteq T_G$ and hence $T_G = \bar{T}_G$

case ii; If $(x,y) \in T_G'$

where T_G' is the set of all limit points of T_G

$\therefore (x,y)$ is a limit point of T_G

Hence $\exists$ a sequence $\{x_n, T(x_n)\}$ in $T_G \ni (x_n, T(x_n)) \to (x,y)$

$\Rightarrow x_n \to x$ & $T(x_n) \to y$

But we have T is continuous & $x_n \to x \Rightarrow T(x_n) \to T(x)$

$\therefore T(x) = y$

Thus, we have $(x,y) = (x, T(x)) \in T_G$

$\therefore \bar{T}_G \subseteq T_G$ — (2)

$\therefore$ (1) and (2) we have $T_G = \bar{T}_G$

Part:-2 Conversely, suppose that T_G is closed.

Let us denote B_1 is a linear space equipped by norm defined

$\|x\|_1 = \|x\| + \|T(x)\|$

Now consider $\|T(x)\| \leq \|T(x)\| + \|x\|$

$= \|x\|_1$

$\Rightarrow \|T(x)\| \leq \|x\|_1$

$\Rightarrow T$ is bounded and hence it is continuous has a mapping B_1 to B' then it is sufficient to prove that B_1 & B are topological.

We have to prove that B_1 and B' are homeomorphic

consider the identity mapping $I_1 : B_1 \to B'$ by $I_1(x) = x \; \forall x \in B_1$

clearly I_1 is one-one & onto.

consider $\|I_1(x)\| = \|x\|$

$\leq \|x\| + \|T(x)\|$

$= \|x\|_1$

$\therefore \|I_1(x)\| \leq 1 \|x\|_1$

$\therefore I_1$ is bounded and hence continuous.

Now we p.T. B_1 is a banach space.

$\because B_1$ is a normed linear space it remains to p.T. B_1 is complete

Consider $\{x_n\}$ is a cauchy sequence in B_1

By def, $\|x_n - x_m\| \to 0$ as $m, n \to \infty$

$\Rightarrow \|x_n - x_m\| + \|T(x_n - x_m)\| \to 0$ as $m, n \to \infty$

$\Rightarrow \|x_n - x_m\| \to 0$ as $\|T(x_n - x_m)\| \to 0$ where $m, n \to \infty$

where $\{x_n\}$ is a cauchy sequence in B_1 and $\{T(x_n)\}$ is a cauchy sequence in B'

Since B_1 and B' are complete $\exists x \in B_1 \ni x_n \to x$ & $\exists y \in B' \ni T(x_n) \to y$

$\because$ The graph of T is closed we have $(x, y) \in T_G \ni T(x) = y$

Now consider $\|x_n - x\|_1$ over $= \|x_n - x\| + \|T(x_n - x)\|$

$= \|x_n - x\| + \|T(x_n) - T(x)\|$

$= \|x_n - x\| + \|T(x_n) - y\| \to 0$ as $n \to \infty$

$\therefore x_n \to x$ in B_1

and hence B_1 is complete.

$\therefore$ By Theorem (B) we have the mapping I_1 is a homeomorphism

$\therefore B_1 = B$ and hence T is cont, linear transformation from B onto B'. Hence proved.

Problem:- If T is a cont. linear transformation of a normed linear space N into a normed linear space N' and if M is its null space. Show that T induces a natural linear transformation $T': \frac{N}{M} \to N'$ and $\|T'\| = \|T\|$.

Proof:- Given that N and N' are two normed linear spaces and $T: N \to N'$ is a cont. linear transformation.

And M is the null space of N.

Claim:- T induces a natural linear transformation $T': \frac{N}{M} \to N'$ and $\|T'\| = \|T\|$

define $T': \frac{N}{M} \to N'$ by $T'(x+M) = T(x)\ \forall\ x \in N$ —(1)

To prove that T is a linear transformation:-

let $(x+M), (y+M) \in \frac{N}{M}$ and any scalars α, β

consider, $T'(\alpha(x+M) + \beta(y+M)) = T'(\alpha x + M + \beta y + M)$

$= T'((\alpha x + \beta y) + M)$

$= T(\alpha x + \beta y)$ ($\because$ by (1))

$= \alpha T(x) + \beta T(y)$

$= \alpha T'(x+M) + \beta T'(y+M)$

To prove that $\|T'\| = \|T\|$:-

consider $\|T'\| = \sup\{|T'(x+M)| : \|x+M\| \leq 1\}$

$= \sup\{|T(x)| : \|x+m\| \leq 1\ \forall\ m \in M\}$

$= \sup\{|T(x+0)| : \|x+m\| \leq 1, \forall\ m \in M\}$

$= \sup\{|T(x)+m| : \|x+m\| \leq 1, \forall\ m \in M\}$

$= \sup\{|T(x+m)| : \|x+m\| \leq 1, \forall\ m \in M\}$

$= \sup\{|T(y)| : \|y\| \leq 1\}$ where $y = x+m$

$= \|T\|$

$\therefore \|T'\| = \|T\|$

Q, If 'M' is a closed linear subspace of a normed linear space and if T is the natural mapping from N onto $\frac{N}{M}$. Define by $x+M$ so that 'T' is a cont. linear transformation for which $\|T\| \leq 1$

Sol:- Given that M is a closed linear subspace of a normed linear space 'N'.

And T is a natural mapping from N onto $\frac{N}{M}$ defined by $T(x)$

claim:- T is a cont. linear transformation & $\|T\| \leq 1$

By theorem (A) in 46^{th} section we have $\frac{N}{M}$ is a normed lin.

ce with respect to the norm defined by $\|x+M\| = \inf\{\|x+m\| : m \in M\}$

To prove that T is a linear transformation:

Let $x, y \in N$ & α, β be any scalars.

consider, $T(\alpha x + \beta y) = (\alpha x + \beta y) + M$

$$= \alpha x + M + \beta y + M$$

$$= \alpha(x+M) + \beta(y+M)$$

$$= \alpha T(x) + \beta T(y)$$

$\therefore T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$

$\therefore T$ is a linear transformation.

To prove that T is continuous:-

To prove this we have to prove that T is bounded.

$\|T(x)\| = \|x+M\|$,

$$= \inf\{\|x+m\| : m \in M\},$$

$$\leq \|x+M\|$$

If $m=0$, $\|T(x)\| \leq \|x\|$

$\|T(x)\| \leq 1 \cdot \|x\|$

$\therefore T$ is bounded and hence T is continuous.

consider $\|T\| = \sup\{\|T(x)\| : \|x\| \leq 1\}$

$$= \sup\{\|x\| : \|x\| \leq 1\}$$

$$\leq 1$$

$\therefore \|T\| \leq 1$